Számítógépvezérelt szabályozások elmélete

Kalman szűrő

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Observer desing for CT-LTI systems

Problem statement

Given:

- a SISO state-space model with parameters (A, B, C)
- a finite **measurement record** of u and y as signals
- an initial value $\hat{x}_0$

Compute:

An estimate of the state signal x over the finite time interval such that $x(t) \to \hat{x}(t)$ as $t \to \infty$

Observer equation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

consider the **observer**

$$\frac{\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) + L(y - C\hat{x}(t))$$

Introduce the **estimation error** signal: $\check{x} = x - \hat{x}$

$$\frac{\check{x}(t)}{dt} = (A - LC)\check{x}(t)$$

If the matrix $\check{A} = A - LC$ is a stability matrix then $\check{x} \to 0$ when $t \to \infty$ (asymptotic stability).

Task: find L such that $\check{A} = A - LC$ is a stability matrix

System model

State-space model of discrete time stochastic LTI systems

$$x(k+1) = \Phi x(k) + \Gamma u(k) + v(k)$$
$$y(k) = Cx(k) + e(k)$$

$$\Phi \in \mathcal{R}^{n \times n} \ , \ \Gamma \in \mathcal{R}^{n \times r} \ , \ C \in \mathcal{R}^{p \times n}$$

and with independent discrete time zero mean Gaussian white noise processes $\{v(k)\}_0^\infty$ and $\{e(k)\}_0^\infty$

$$E[v(k)v^T(k)] = R_1 \ , \ E[v(k)v^T(j)] = 0 \ , \ \forall\ k \neq j \ , \ E[v(k)e^T(j)] = 0 \ , \ \forall\ k, j$$
$$E[e(k)e^T(k)] = R_2 \ , \ E[e(k)e^T(j)] = 0 \ , \ \forall\ k \neq j$$

Initial conditions

$$Ex(0) = m_0 \ , \ cov[x(0)] = R_0$$

Parameters:

$$(\Phi, \Gamma, C \ ; \ R_1, R_2 \ ; \ m_0, R_0)$$

Prediction, filtering and smoothing

Problem statement

Given:

- a finite *measurement record* about a SISO system consisting of a finite set of measured input and output (deterministic) values at the discrete time instances

$$D(0,k) = \{y(i), u(i) \mid i = 0, 1, \ldots, k\} \quad (1)$$

- the state space model model of a discrete time stochastic system

Estimate:
the value of the state $\hat{x}(k+m)$ based on the measurement record available.

Depending on the value of m we have the following special cases:

- $m < 0$ is termed *smoothing*,
- $m = 0$ is termed *filtering*
- $m > 0$ is *prediction*

Kalman-filter: the problem

Given
State estimator (**Kalman estimator**) in the form

$$\hat{x}(k+1|k) = \Phi\hat{x}(k|k-1) + \Gamma u(k) + K(k)\left[y(k) - C\hat{x}(k|k-1)\right] \quad , \quad E\hat{x}(0) = m_0$$

for the discrete time SISO LTI stochastic system given by a SSR.
The *reconstruction error* $\tilde{x}(k) = x(k) - \hat{x}(k)$

$$\begin{aligned} \tilde{x}(k+1) &= \Phi\tilde{x}(k) + v(k) - K(k)[y(k) - C\hat{x}(k|k-1)] \\ \tilde{x}(k+1) &= (\Phi - K(k)C)\tilde{x}(k) + v(k) - K(k)e(k) \qquad (*) \end{aligned}$$

Covariance function $P(k)$ of the reconstruction error process:

$$P(k) = E\{\ [\tilde{x}(k) - E\tilde{x}(k)]\ [\tilde{x}(k) - E\tilde{x}(k)]^T\ \}$$

Find
optimal $K(k)$ which minimizes the scalar $\alpha^T P(k+1)\alpha$

Reconstruction error process 1

Mean value

From

$$\tilde{x}(k+1) = (\Phi - K(k)C)\tilde{x}(k) + v(k) - K(k)e(k)$$

we obtain for the mean value of the reconstruction error

$$E\tilde{x}(k+1) = (\Phi - K(k)C)E\tilde{x}(k)$$

with $E\tilde{x}(0) = 0$ as initial condition. Therefore

$$E\tilde{x}(k) = 0 \quad \textit{for all} \quad k \geq 0$$

that is the **Kalman estimator is unbiased in statistical sense.**

Reconstruction error process 2

Covariance function

Again

$$\tilde{x}(k+1) = (\Phi - K(k)C)\tilde{x}(k) + v(k) - K(k)e(k)$$

can be used for computing the covariance function $P(k)$ of the reconstruction error process

$$\begin{aligned} P(k+1) &= E[\tilde{x}(k+1)\tilde{x}^T(k+1)] = \\ &= (\Phi - K(k)C)P(k)(\Phi - K(k)C)^T + R_1 + K(k)R_2K^T(k) \end{aligned}$$

with $P(0) = P_0$.

From the above equation we see that **if $P(k)$ is positive semidefinite then $P(k+1)$ is also positive semidefinite**.

Lemma: Completing the squares

Minimize the scalar function

$$F(u) = u^T S u + r^T u + u^T r$$

with S being a symmetric positive definite matrix, $u, r \in \mathcal{R}^n$ and $S \in \mathcal{R}^{n \times n}$.
Let us complete the squares in the function above as follows:

$$F(u) = u^T S u + r^T u + u^T r + r^T S^{-1} r - r^T S^{-1} r$$
$$F(u) = (u + S^{-1} r)^T S (u + S^{-1} r) - r^T S^{-1} r$$

From the form above we can see that F is minimized by

$$u = -S^{-1} r \quad \textit{and} \quad F_{min} = -r^T S^{-1} r$$

Computation of the Kalman-gain 1

Compute the minimum of the scalar $\alpha^T P(k+1)\alpha$

$$\begin{aligned}
&\alpha^T P(k+1)\alpha = \\
&= \alpha^T\{\Phi P(k)\Phi + R_1 + K(k)(R_2 + CP(k)C^T)K^T(k) - \\
&- K(k)CP(k)\Phi^T - \Phi P(k)C^T K^T(k)\}\alpha
\end{aligned}$$

Separate the constant and Kalman-gain dependent terms

$$\begin{aligned}
&\alpha^T P(k+1)\alpha = \alpha^T\{\Phi P(k)\Phi + R_1\}\alpha + \\
&+ \alpha^T\{K(k)(R_2 + CP(k)C^T)K^T(k) - K(k)CP(k)\Phi^T - \Phi P(k)C^T K^T(k)\}\alpha
\end{aligned}$$

Computation of the Kalman-gain 2

$K(k)$ can be selected to minimize

$$\alpha^T\{K(k)(R_2 + CP(k)C^T)K^T(k) - K(k)CP(k)\Phi^T - \Phi P(k)C^T K^T(k)\}\alpha$$

using the "completing of the squares" lemma with

- $u = K^T(k)\alpha$,
- $r = -CP(k)\Phi^T\alpha$ and
- $S = R_2 + CP(k)C^T$

The minimizing u is (with **arbitrary** α)

$$u = K^T(k)\alpha = (R_2 + CP(k)C^T)^{-1}CP(k)\Phi^T\alpha$$

Computation of the Kalman-gain 3

The optimal gain

$$K(k) = \Phi P(k) C^T (R_2 + C P(k) C^T)^{-1}$$

The covariance function of the reconstruction error

$$P(k+1) = \Phi P(k) \Phi^T + R_1 - \Phi P(k) C^T (R_2 + C P(k) C^T)^{-1} C P(k) \Phi^T$$

Remarks

1. The filter structure is optimal for Gaussian sequences only.
2. $K(k)$ and $P(k)$ can be pre-computed. Then only the filter equation is used in each step.
3. The filter is very sensitive with respect to the deviation from LTI system model and Gaussian white noise state and output error processes.