Számítógépvezérelt szabályozások elmélete

Szabályozótervezés

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2011. március 29.

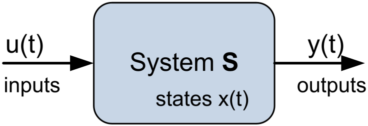
Overview

Systems

- System (**S**): acts on signals

$$y = \mathbf{S}[u]$$

  - inputs (u) and outputs (y)



CT-LTI I/O system models – 3

- Operator domain I/O model for SISO systems
- Transfer function

$$Y(s) = H(s)U(s)$$

with

$Y(s)$	Laplace-transform of the output signal
$U(s)$	Laplace-transform of the input signal
$H(s) = \frac{b(s)}{a(s)}$	*transfer function of the system*
	where $a(s)$ and $b(s)$ are polynomials and
	degree $b(s) = m$
	degree $a(s) = n$, characteristic polynomial of A

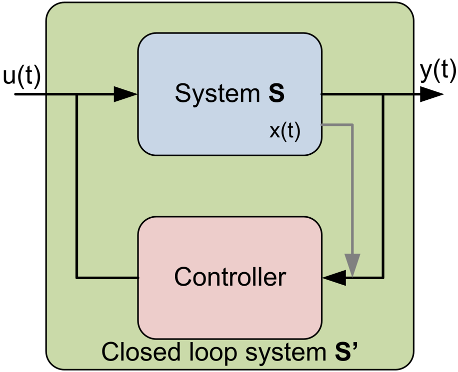
- Asymptotically stable *iff* the roots of $a(s)$ have negative real parts

Control – general problem statement

- Given
  - a system model
  - *control goal*
- Compute an *input record* to fulfil the control goal

  Control goals:
  - stabilization
  - disturbance rejection
  - optimal control

Control – signal flow diagram



Feedback

- *state feedback* when the input depends only on states, i.e.

$$u = F(x)$$

- *output feedback* when the input depends only on outputs, i.e.

$$u = F(y)$$

- *static feedback* when the function F is static,
- *linear static feedback* when the function F is a linear static function,
- *full state feedback* when the input signal depends on *every element in the state vector*.

Overview

General problem statement

- Given
  - a *SISO LTI* system with realization (A, B, C) (poles are determined by $a(s)$)
  - specified (desired) poles by a polynomial $\alpha(s)$ such that $deg\ a(s) = deg\ \alpha(s) = n$
- Compute
  - a *full state feedback* such that the poles of the closed loop systems are the roots of $\alpha(s)$

A subproblem of this problem statement is to find a feedback to *stabilize* the system.

Closed-loop LTI systems

- (A, B, C) of a SISO LTI system **S**

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) \\
y(t)\ ,\ u(t) &\in \mathbb{R} \quad , \quad x(t) \in \mathbb{R}^n \\
A \in \mathbb{R}^{n\times n}\ &,\ B \in \mathbb{R}^{n\times 1}\ ,\ C \in \mathbb{R}^{1\times n}
\end{aligned}$$

- and linear static full state feedback

$$\begin{aligned}
& u(t) = v(t) - k \cdot x(t) \quad (v(t) = u(t) + k \cdot x(t)) \\
& k = \left[\begin{array}{cccccc} k_1 & k_2 & . & . & . & k_n \end{array}\right] \\
& k \in \mathbb{R}^{1\times n} \qquad (row\ \ vector)
\end{aligned}$$

- Closed-loop system model

$$\begin{aligned}
\dot{x}(t) &= (A - Bk)x(t) + Bv(t) \\
y(t) &= Cx(t)
\end{aligned}$$

- Characteristic polynomials

$$a_c(s) = det\ (sI - A + Bk) \quad := \alpha(s) \quad , \quad a(s) = det\ (sI - A)$$

Bass-Gura formula

- Determinant of triangular block matrices

$$det\begin{bmatrix} P & 0 \\ R & S \end{bmatrix} = det\begin{bmatrix} P & Q \\ 0 & S \end{bmatrix} = det(P)\cdot det(S)$$

- Full block matrices can be decomposed to product os lower and upper triangular matrices

$$\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} P & 0 \\ R & I \end{bmatrix}\cdot\begin{bmatrix} I & P^{-1}Q \\ 0 & S - RP^{-1}Q \end{bmatrix}$$

$$\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} I & Q \\ 0 & S \end{bmatrix}\cdot\begin{bmatrix} P - QS^{-1}R & 0 \\ S^{-1}R & I \end{bmatrix}$$

$$det(P)\cdot 1\cdot 1\cdot det(S - RP^{-1}Q) = 1\cdot det(S)\cdot det(P - QS^{-1}R)\cdot 1$$

- Apply it for

$$det\begin{bmatrix} sI - A & B \\ -k & 1 \end{bmatrix}$$

- and we get

$$det(sI - A)det(1 + k(sI - A)^{-1}B) = 1det((sI - A) + B1^{-1}k)$$

Pole placement – 1

$$det(sI-A)det(1+k(sI-A)^{-1}B) = 1det((sI-A)+B1^{-1}k)$$

$$a(s)(1+k(sI-A)^{-1}B) = det(sI-A+Bk) = \alpha(s)$$

$$\alpha(s) = a(s)(1+k(sI-A)^{-1}B) \quad \Rightarrow \quad \alpha(s)-a(s) = a(s)k(sI-A)^{-1}B$$

With the *resolvent formula*

$$(sI-A)^{-1} = \frac{1}{a(s)}(s^{n-1}I + s^{n-2}(A+a_1I) + s^{n-3}(A^2+a_1A+a_2I) + \ldots$$

we get

$$(\alpha_1-a_1)s^{n-1} + (\alpha_2-a_2)s^{n-2} + \ldots + (\alpha_n-a_n) = \\ = kBs^{n-1} + k(A+a_1I)Bs^{n-2} + \ldots$$

Pole placement – 2

$$(\alpha_1 - a_1)s^{n-1} + (\alpha_2 - a_2)s^{n-2} + \ldots + (\alpha_n - a_n) = kBs^{n-1} + k(A + a_1 I)Bs^{n-2} + \ldots$$

a *polynomial equation*

$$\begin{aligned}
\alpha_1 - a_1 &= kB \\
\alpha_2 - a_2 &= kAB + a_1 kB = a_1 kB + kAB \\
\alpha_3 - a_3 &= kA^2B + a_1 kAB + a_2 kB = a_2 kB + a_1 kAB + kA^2 B \\
&. \\
&.
\end{aligned}$$

$$\underline{\alpha} - \underline{a} = k \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 & . & . & . & a_{n-1} \\ 0 & 1 & a_1 & . & . & . & a_{n-2} \\ 0 & 0 & 1 & . & . & . & a_{n-3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

Pole placement controller

$$\underline{\alpha} - \underline{a} = k \left[\begin{array}{ccccc} B & AB & A^2B & \ldots & A^{n-1}B \end{array} \right] \begin{bmatrix} 1 & a_1 & a_2 & . & . & . & a_{n-1} \\ 0 & 1 & a_1 & . & . & . & a_{n-2} \\ 0 & 0 & 1 & . & . & . & a_{n-3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}$$

$$\underline{\alpha} - \underline{a} = k\mathcal{C} T_\ell^T$$

- If **S** is *controllable* then

$$k = (\underline{\alpha} - \underline{a}) T_\ell^{-T} \mathcal{C}^{-1}$$

- Keep in mind: $\underline{\alpha}$ and $\underline{a}$ are coefficient vectors, not the vectors of roots!

Pole placement in controller form realization

- Recall that

$$\dot{x}(t) = A_c x(t) + B_c u(t)$$
$$y(t) = C_c x(t)$$

with

$$A_c = \begin{bmatrix} -a_1 & -a_2 & . & . & . & -a_n \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} b_1 & b_2 & . & . & . & b_n \end{bmatrix}$$

- with the coefficients of the polynomials $a(s) = s^n + a_1 s^{n-1} + ... + a_{n-1}s + a_n$ and $b(s) = b_1 s^{n-1} + ... + b_{n-1}s + b_n$ that appear in the transfer function $H(s) = \frac{b(s)}{a(s)}$

Pole placement in controller form realization

$$A_c - B_c k_c = \begin{bmatrix} -(a_1 + k_{c1}) & -(a_2 + k_{c2}) & . & . & . & -(a_n + k_{cn}) \\ 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}$$

- Closed-loop characteristic polynomial $\alpha(s)$ is

$$\alpha(s) = det(sI - A_c + B_c k_c) = s^n + (a_1 + k_{c1})s^{n-1} + ... + (a_n + k_{cn})$$

- State feedback coefficients in k_c are

$$k_c = \underline{\alpha} - \underline{a}$$

Overview

LQR: problem statement

- Given
  - a (MIMO) LTI state space model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad , \quad x(0) = x_0 \\ y(t) &= Cx(t) \end{aligned}$$

$$\begin{aligned} &y(t) \in \mathbb{R}^p \ , \ u(t) \in \mathbb{R}^r \ , \ x(t) \in \mathbb{R}^n \\ &A \in \mathbb{R}^{n\times n} \ , \ B \in \mathbb{R}^{n\times r} \ , \ C \in \mathbb{R}^{p\times n} \end{aligned}$$

  - a functional (control aim)

$$J(x,u) = \frac{1}{2}\int_0^T [x^T(t)Qx(t) + u^T(t)Ru(t)]dt$$

  with $Q^T = Q, \ Q > 0$ and $R^T = R, \ R > 0$.

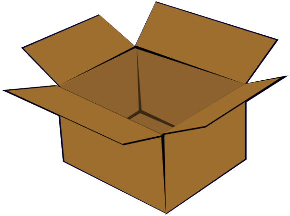
- Compute
  - a control $\{u(t) \ , \quad t \in [0, T]\}$ that minimizes J subject to the state-space model.

Constrained optimization using Lagrange multipliers

- Aim
  Optimize (minimize or maximize)
  $$f(x, y, z)$$
  with respect to x, y, z subject to the constraint $g(x, y, z) = k$.
- Solution:
  - Study the Lagrange function's stationary points
    $$\Lambda(x, y, z, \lambda) = f(x, y, z) + \lambda \cdot (g(x, y, z) - k)$$
  - solve the following (system of) equations
    $$\begin{aligned} \nabla f(x, y, z) &= -\lambda \nabla g(x, y, z) \\ g(x, y, z) &= k \end{aligned}$$
  - λ - Lagrange multiplier
  - four variables (x, y, z, λ)

Example

Find the dimensions of the box with largest volume if the total surface area is 64 cm^2*!*



- The volume is the box is given by

$$f(x,y,z) = xyz$$

- The surface constraint is given by

$$2xy + 2xz + 2yz = 64 \quad \Leftrightarrow \quad xy + xz + yz = g(x,y,z) = 32$$

- Four equations to be solved

$$\begin{array}{rclcrcl} \frac{\partial f}{\partial x} & = & -\lambda \frac{\partial g}{\partial x} & \Leftrightarrow & yz & = & -\lambda(y+z) \\ \frac{\partial f}{\partial y} & = & -\lambda \frac{\partial g}{\partial y} & \Leftrightarrow & xz & = & -\lambda(x+z) \\ \frac{\partial f}{\partial z} & = & -\lambda \frac{\partial g}{\partial z} & \Leftrightarrow & xy & = & -\lambda(x+y) \\ g(x,y,z) & = & k & \Leftrightarrow & xy+xz+yz & = & 32 \end{array}$$

- Solution is $x = y = z = \sqrt{\frac{32}{3}}$ cm

Calculus of variations – 1

- Problem statement: Minimize

$$J(x,u) = \int_0^T F(x,u,t)dt$$

with respect to u subject to $\dot{x} = f(x,u,t)$.

- Solution:
  - by using a vector Lagrange multiplier $\lambda(.)$

$$J(x,\dot{x},u) = \int_0^T [F(x,u,t) + \lambda^T(t)(f(x,u,t) - \dot{x})]dt$$

  - and the Hamiltonian function $H = F + \lambda^T f$.

$$J = \int_0^T [H - \lambda^T \dot{x}]dt$$

Calculus of variations – 2

- $\dot{x}$ is eliminated by integrating in part using

$$[\lambda^T x]_0^T = \int_0^T \dot{\lambda}^T x + \int_0^T \lambda^T \dot{x}$$

- then $J = \int_0^T [H - \lambda^T \dot{x}] dt$ transforms to

$$J = \int_0^T [H + \dot{\lambda}^T x] dt - [\lambda^T x]_0^T$$

- If a minimizing u were found, arbitrary δu in u and δx in x should produce $\delta J = 0$

$$\delta J = -\lambda^T \delta x|_0^T + \int_0^T \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \delta x + \frac{\partial H}{\partial u} \delta u \right] dt$$

Euler-Lagrange equations

- δJ is zero in

$$\delta J = -\lambda^T \delta x|_0^T + \int_0^T \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \delta x + \frac{\partial H}{\partial u} \delta u \right] dt$$

when the Euler-Lagrange equations hold

$$\frac{\partial H}{\partial x} + \dot{\lambda}^T = 0 \quad , \quad \frac{\partial H}{\partial u} = 0$$

- with the Hamiltonian

$$H = F + \lambda^T f$$

Euler-Lagrange equations in the LQR case

- Euler-Lagrange equations with the Hamiltonian $H = F + \lambda^T f$:

$$\frac{\partial H}{\partial x} + \dot{\lambda}^T = 0 \quad , \quad \frac{\partial H}{\partial u} = 0$$

- Special problem elements:

$$\begin{aligned} f &= Ax + Bu \\ F &= \tfrac{1}{2}(x^T Q x + u^T R u) \\ H &= \tfrac{1}{2}(x^T Q x + u^T R u) + \lambda^T (Ax + Bu) \end{aligned}$$

- LQR Euler-Lagrange equations: with $\frac{\partial}{\partial x}(x^T Q x) = 2x^T Q$

$$\dot{\lambda}^T + x^T Q + \lambda^T A = 0 \quad , \quad \lambda^T(T) = 0$$

$$u^T R + \lambda^T B = 0$$

State and co-state dynamics

- Rearranged LQR Euler-Lagrange equations (Q and R are symmetric matrices)

$$\begin{aligned}\dot{\lambda}(t) &= -Qx(t) - A^T\lambda(t)\ , \quad \lambda(T) = 0 \\ u(t) &= -R^{-1}B^T\lambda(t)\end{aligned}$$

- State equation:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(0) = x_0$$

- Joint matrix-vector form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \quad , \quad \begin{matrix} x(0) = x_0 \\ \lambda(T) = 0 \end{matrix}$$

- System dynamics + Hammerstein co-state diff. eq.

LQR: Controllable and Observable case

Lemma

When (A, B) is controllable and (C, A) is observable

$$\lambda(t) = K(t)x(t) \quad , \quad K(t) \in \mathcal{R}^{n \times n}$$

- The modified state and co-state equations

$$\begin{aligned}
\dot{\lambda}(t) = -Qx(t) - A^T\lambda(t) \quad &\Rightarrow \quad \dot{K}(t)x(t) + K(t)\dot{x}(t) = -A^TK(t)x(t) - Qx(t) \\
u(t) = -R^{-1}B^T\lambda(t) \quad &\Rightarrow \quad u(t) = -R^{-1}B^TK(t)x(t) \\
\dot{x}(t) = Ax(t) + Bu(t) \quad &\Rightarrow \quad \dot{x}(t) = Ax(t) - BR^{-1}B^TK(t)x(t)
\end{aligned}$$

$$\dot{K}(t)x(t) + K(t)[A - BR^{-1}B^TK(t)]x(t) + A^TK(t)x(t) + Qx(t) = 0$$

for any $x(t)$.

- Matrix Riccati Differential Equation for $K(t)$

$$\dot{K}(t) + K(t)A + A^TK(t) - K(t)BR^{-1}B^TK(t) + Q = 0$$

Stationary case

- Special case: stationary solution with $T \to \infty$

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt$$

$$\lim_{t \to \infty} K(t) = K \quad \text{i.e.} \quad \dot{K}(t) = 0$$

- Control Algebraic Riccati Equation (CARE)

$$KA + A^T K - KBR^{-1}B^T K + Q = 0$$

Theorem (R. Kalman)

If (C, A) is observable and (A, B) is controllable then CARE has a unique positive definite symmetric solution K.

LQR and its properties

- Solution: *linear static full state feedback*

$$u^0(t) = -R^{-1}B^T Kx(t) = -Gx(t)$$

where $G = R^{-1}B^T K$.

- *Closed loop dynamics*

$$\dot{x} = Ax - BR^{-1}B^T Kx = (A - BG)x \quad , \quad x(0) = x_0$$

- Properties of the closed-loop system
  - the closed-loop system is asymptotically stable no matter what the values of A, B, C, R, Q are, i.e.

$$Re\ \lambda_i(A - BG) < 0 \quad , \quad i = 1, 2, ..., n$$

  - specific location of the closed-loop poles depend on the choice of Q and R

Overview

Servo control: problem statement

- Aim: to follow a time-dependent reference signal $r(t)$
- Given : the state equation of an *extended* LTI system model

$$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad x(0) = x_0$$
$$\dot{z}(t) = r(t) - y(t) = r(t) - Cx(t)$$

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

In steady-state $\dot{z} = 0$, i.e. $r = y$ or $r = Cx$.

- Compute a stabilizing feedback

$$u = -[K_x K_z] \cdot \begin{bmatrix} x \\ z \end{bmatrix}$$

Servo control: solution

- Control gain design: by using pole-placement or LQR design procedure with the extended system parameter matrices

$$A' = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \quad , \quad B' = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

- Applicability condition: (A', B') should be a controllable pair